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RESEARCH MEMORANDUM

ON WEAK CONVERGENCE OF STRATEGIES IN CERTAIN GAMES OVER A FUNCTION SPACE

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Summary: For a certain class of games over a function space, mixed strategies are weak limits of pure strategies. It follows that, except in special instances, no compact topology can be imposed on the pure strategy spaces for a game of this type in such a way that the payoff is continuous, even when the payoff has a continuous kernel and the pure strategy spaces are bounded.

ON WEAK CONVERGENCE OF STRATEGIES IN CERTAIN GAMES OVER A FUNCTION SPACE

Wendell Fleming

1. We consider here 2-person zero sum games which have as pure strategies bounded measurable functions x and y on the closed unit interval $(0,1)$ and payoffs of the form

$$\text{(a)} \quad M(x,y) = \int_0^1 K[x(t),y(t)]v(t)dt,$$

where K is continuous and v is summable.

We write $f:x_\theta$, $0 \leq \theta \leq 1$, to denote the strategy for the maximizing player which chooses the function x_θ according to the uniform distribution on θ . If $x_\theta(t)$ is measurable in (t,θ), such an f is called a <u>1-parameter mixed strategy</u>[1]. In the same way, $g:y_\phi$ denotes a 1-parameter mixed strategy for the minimizing player. For such strategies f, g the expectation is given by

$$\text{(b)} \quad M(f,g) = \int_0^1 \int_0^1 \int_0^1 K[x_\theta(t),y_\phi(t)]v(t)dt\, d\theta d\phi \ .$$

[1] It should be noted that no generality would be gained here by admitting families of functions x_θ distributed on $(0,1)$ according to an arbitrary distribution $F(\theta)$. For, such a distribution reduces to a uniform distribution by the change of variables $\theta = F^{-1}(\tau)$, $\bar{x}_\tau = x_{F^{-1}(\tau)}$, where $F^{-1}(\tau) = \sup_s [\tau \geq F(s)]$.

<u>Theorem</u>: Given a 1-parameter mixed strategy $f: x_\theta$ for the maximizing player with $0 \leq x_\theta(t) \leq 1$ for all t and θ, there exists a sequence (X_n) of step functions with $0 \leq X_n(t) \leq 1$ for all t and $n = 1,2,\ldots$ such that

$$\lim_{n \to \infty} M(X_n, y) = M(f,y)$$

for every bounded measurable function y and every payoff M of the form (a) with continuous kernel K. If all the x_θ satisfy the side condition $\int_0^1 x_\theta(t)dt = A$, then the X_n may be chosen so as to satisfy this same condition.

The corresponding statement is true for the minimizing player.

The theorem above states that the set of pure strategies is "weakly" dense in the set of 1-parameter mixed strategies. No uniformity of convergence is asserted; thus, the theorem does <u>not</u> imply the statement (which is, of course, false for many payoffs) that pure strategies are "almost as good" as mixed strategies, i.e., that $\max_x \min_y = \max_f \min_y$.

Our theorem is an easy consequence of a theorem on linear averages due to L. C. Young [3, p.225]. We leave the proof to § 2, and consider first an interesting corollary.

Let G be a game with payoff $M(x,y)$. Two mixed strategies f_1, f_2 for the maximizing player are called <u>equivalent (G)</u> if $M(f_1,y) = M(f_2,y)$ for all his opponent's pure strategies y.

For each A, let S_A denote the set of all measurable functions x which satisfy $0 \leq x(t) \leq 1$ and $\int_0^1 x(t)dt = A$.

Seeking an analogy between games over the unit square with continuous payoffs and the games considered here, one might look for topologies in which the spaces S_A are compact and the payoffs M of type (a) are continuous. In most instances no such

topologies exist, as is shown by:

<u>Corollary</u>: Let G be a game which has payoff M of form (a) with continuous kernel K and S_A as the pure strategy space for the maximizing player. Then a necessary condition that a compact topology T on S_A exist with the property that $M(x,y)$ is a continuous function of x on S_A for each pure strategy y for the minimizing player is that every 1-parameter mixed strategy $f:x_\theta$, with x_θ in S_A for all θ, be (G) equivalent to some pure strategy x_0 in S_A.

In particular, if the maximizing player has an optimal 1-parameter mixed strategy but no optimal pure strategy, this optimal strategy is (G) equivalent to no pure strategy, and the necessary condition in the corollary does not hold.[1]

To prove the corollary, let $\sum$ denote the space of pure strategies for the minimizing player. Suppose that a 1-parameter mixed strategy $f:x_\theta$ with all x_θ in S_A exists such that for no x in S_A does one have $M(f,y) = M(x,y)$ for all y in $\sum$. By the Theorem, a sequence (X_n) in S_A exists with the property that $M(f,y) = \lim_{n\to\infty} M(X_n,y)$ for every bounded measurable function y.

Suppose that a topology T on S_A exists in which S_A is compact and M is continuous in x for each y in $\sum$. Since S_A is compact (T), a directed subsequence (X_{n_α}) of the sequence (X_n) converges (T) to some x_0 in S_A.[2] By the (T) continuity of M in x for each y in $\sum$,

$$M(f,y) = \lim_{n\to\infty} M(X_n,y) = \lim_{\alpha} M(X_{n_\alpha},y) = M(x_0,y)$$

for all y in $\sum$, a contradiction.

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1 It is known that many games with payoffs of type (a) have optimal 1-parameter mixed strategies (cf. [1]).

2 Since it is not assumed that T satisfies any countability axioms, it is necessary to use directed (i.e., non-countable) sequences here.

2. The theorem stated in §1 is proved in this section. The portions used here of Young's proof of the result [3,p.225] cited previously are stated as Lemmas 1 and 2.

Lemma 1; Let $h(t)$ be continuous in an interval (a,b), and let $p_1,\dots,p_m$ be constants with $p_i > 0$ for all i and $\sum_{i=1}^{m} p_i = 1$. For each $n = 1,2,\dots,$ subdivide (a,b) into n equal intervals $\Delta_n^1,\dots,\Delta_n^n$ and further subdivide each Δ_n^j into m intervals $\Delta_n^{1j},\dots,\Delta_n^{mj}$ in such a way that

$\frac{|\Delta_n^{ij}|}{|\Delta_n^j|} = p_i$ for all $i = 1,2,\dots,m$.[1] Then

$$\lim_{n\to\infty} \sum_{j=1}^{n} \int_{\Delta_n^{ij}} h(t)dt = p_i \int_a^b h(t)dt$$

for all $i = 1,2,\dots,m$.

The proof of Lemma 1 is elementary, and hence is left to the reader.

Lemma 2: Let $x_1,\dots,x_m$ and $p_1,\dots,p_m$ be constants with $p_i > 0$ for all i and $\sum_{i=1}^{m} p_i = 1$. Then given an interval (a,b), a sequence (X_n) of step functions defined on (a,b) and assuming only the values $x_1,\dots,x_m$ exists such that $\int_a^b X_n(t)dt = (b-a) \sum_{i=1}^{m} p_i x_i$ for each $n = 1,2,\dots$ and

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[1] (Δ) denotes the length of an interval Δ .

$$\lim_{n\to\infty}\int_a^b \phi(X_n(t),t)dt = \sum_{i=1}^m p_i\int_a^b \phi(x_i,t)dt$$

for every continuous function $\phi(x,t)$.

Proof: Subdivide (a,b) as in Lemma 1. Define $X_n(t)$ to be x_i for those t which are in intervals Δ_n^{ij}, $j = 1,2,\ldots,n$. Apply Lemma 1 to the functions $h_i(t) = \phi(x_i,t)$ and sum with respect to i.

To prove the main theorem, let $\phi_1,\ldots,\phi_n,\ldots$ be a denumerable set of continuous functions on the unit square such that each continuous function on the unit square is a uniform limit of a sequence of these ϕ_j.

Since $x_\theta(t) = x(t,\theta)$ is measurable, and hence is an almost uniform limit of step functions,[1] for $n = 1,2,\ldots$ a step function $x_n(t,\theta)$ exists with $0 \le x_n(t,\theta) \le 1$ and

$$\int_0^1\int_0^1 x_n(t,\theta)dt \quad d\theta = \int_0^1\int_0^1 x(t,\theta)dt \quad d\theta \quad \text{such that}$$

$$\left|\int_0^1\int_0^1 \left[\phi_j[x(t,\theta),t] - \phi_j[x_n(t,\theta),t]\right] dt \quad d\theta\right| < \frac{1}{n}$$

for $j = 1,2,\ldots,n$.

Since $x_n(t,\theta)$ is a step function, $\int_0^1\int_0^1 \phi[x_n(t,\theta),t]dtd\theta$ may be written in the form

$$\sum_s \sum_i \int_{t_{s-1}}^{t_s} p_{is}\phi(x_{is},t)dt ,$$

where the (t_{s-1},t_s) are non-overlapping intervals in each of

[1] Cf. McShane [2,p.160] for the definition of almost uniform convergence.

which $x_n(t,\theta)$ is constant as a function of t. Applying Lemma 2 in each interval (t_{s-1}, t_s), one gets a sequence $(X_{nk}(t))$ of step functions with $0 \leq X_{nk}(t) \leq 1$ such that

$$\int_{t_{s-1}}^{t_s} X_{nk}(t)\,dt = (t_s - t_{s-1}) \sum_i p_{is} x_{is} = \int_0^1 \int_{t_{s-1}}^{t_s} x_n(t,\theta)\,dt\,d\theta$$

for all $k = 1,2,\ldots$ and

$$\lim_{k \to \infty} \int_{t_{s-1}}^{t_s} \phi[X_{nk}(t),t]\,dt = \sum_i p_{is} \int_{t_{s-1}}^{t_s} \phi[x_{is},t]\,dt =$$

$$\int_0^1 \int_{t_{s-1}}^{t_s} \phi[X_n(t,\theta),t]\,dt\,d\theta$$

for all continuous $\phi(x,t)$.

Summing with respect to s, one gets

$$\lim_{k\to\infty} \int_0^1 \int_0^1 \phi[X_{nk}(t),t]\,dt = \int_0^1 \int_0^1 \phi[x_n(t,\theta),t]\,dt\,d\theta$$

and $\int_0^1 X_{nk}(t)\,dt = \int_0^1 \int_0^1 x_n(t,\theta)\,dt\,d\theta$ for all $k = 1,2,\ldots$.

Hence, for each n, a step function X_n $(= X_{nk_n}$, for some $k_n)$ exists with $0 \leq X_n(t) \leq 1$ and

$\int_0^1 X_n(t)\,dt = \int_0^1 \int_0^1 x(t,\theta)\,dt\,d\theta$ such that

$$\left| \int_0^1 \int_0^1 \phi_j[x(t,\theta),t]\,dt\,d\theta - \int_0^1 \phi_j[X_n(t),t]\,dt \right| < \frac{1}{n}$$

for $j = 1,2,\ldots,n$. Then

$$\lim_{n\to\infty} \int_0^1 \phi_j[X_n(t),t]\,dt = \int_0^1 \int_0^1 \phi_j[x(t,\theta]\,dt\,d\theta$$

for every $j = 1,2,\ldots$. Since the set of ϕ_j 's is uniformly dense, it follows that

$$\lim_{n\to\infty} \int_0^1 \phi[X_n(t),t]\,dt = \int_0^1 \int_0^1 \phi[x(t,\theta),t]\,dt\,d\theta$$

for every continuous $\phi(x,t)$.

If y, K, and v are all continuous functions, $\phi(x,t) = K[x,y(t)]v(t)$ is continuous, and so

$$\text{(c)} \lim_{n\to\infty} \int_0^1 K[X_n(t),y(t)]v(t)\,dt = \int_0^1 \int_0^1 K[x(t,\theta),y(t)]v(t)\,dt.$$

It follows easily from the facts that every continuous kernel K is bounded and uniformly continuous for (x,y) in a bounded portion of the plane, and that every summable function on $(0,1)$ is an almost uniform limit of continuous functions that (c) still holds if we suppose merely that y is bounded and measurable and v is summable. This proves the theorem.

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